

Projections of self-affine sets onto lines

Antti Käenmäki

Warsaw, 22 May 2026

This talk exhibits a recent work
with Balázs Bárány and István Kolossváry

Marstrand–Mattila projection theorem

For $V \in \mathbb{R}\mathbb{P}^{d-1}$, let $\text{proj}_V: \mathbb{R}^d \rightarrow V$ be the orthogonal projection.

Theorem (Marstrand (1954) and Mattila (1975)). *For every Borel set $X \subseteq \mathbb{R}^d$ and $\gamma_{d,1}$ -almost every $V \in \mathbb{R}\mathbb{P}^{d-1}$,*

$$\dim_{\text{H}}(\text{proj}_V(X)) = \min\{1, \dim_{\text{H}}(X)\}.$$

Mattila–Kaufman (1975) proved that the Hausdorff dimension \dim_{H} of the exceptional set can be as large as possible. Therefore, for sets without internal symmetries this is the whole story.

Guiding question. For **structured** sets X , can we describe the *exceptional set*

$$\mathcal{E}(X) = \{V \in \mathbb{R}\mathbb{P}^{d-1} : \dim_{\text{H}}(\text{proj}_V(X)) < \min\{1, \dim_{\text{H}}(X)\}\}?$$

Self-affine sets and irreducibility

An *affine iterated function system* on \mathbb{R}^d is a tuple $\Phi = (\varphi_1, \dots, \varphi_N)$ with maps $\varphi_i(x) = A_i x + t_i$, where $A_i \in \text{GL}(d, \mathbb{R})$ and $\|A_i\| < 1$. There is a unique non-empty compact $X \subset \mathbb{R}^d$ such that

$$X = \bigcup_{i=1}^N \varphi_i(X).$$

We write $A = (A_1, \dots, A_N)$ for the tuple of linear parts and $A_i = A_{i_1} \cdots A_{i_n}$ for the product associated to a word $i = i_1 \cdots i_n$.

Definition. We say that $A \in \text{GL}(d, \mathbb{R})^N$ is

- ▶ *reducible* if there is a proper non-trivial linear subspace $V \subset \mathbb{R}^d$ with $A_i V = V$ for all i ,
- ▶ *strongly irreducible* if no finite union of proper non-trivial linear subspaces of \mathbb{R}^d is invariant under every A_i ,
- ▶ *proximal* if some A_i has a simple dominant eigenvalue.

Exceptional sets in \mathbb{R}^2 : state of the art

Falconer–Kempton (2017). For many dominated self-affine sets $X \subset \mathbb{R}^2$, $\#\mathcal{E}(X) \leq 1$.

Bárány–Hochman–Rapaport (2019). For proximal, strongly irreducible self-affine sets $X \subset \mathbb{R}^2$ satisfying the strong open set condition, $\mathcal{E}(X) = \emptyset$.

Hochman–Rapaport (2021). The strong open set condition in the result above can be relaxed to exponential separation.

Bárány–Käenmäki–Pyörälä–Wu (2023+) and Pyörälä (2025). For self-affine measures on planar carpets, at most two exceptional directions; Pyörälä removes the separation assumption.

Exceptional sets in \mathbb{R}^d : state of the art

Feng–Xie (2025) and Morris–Sert (2025). Under **irreducibility**, $\mathcal{E}(X) = \emptyset$ for **Lebesgue-almost every** choice of translations t_1, \dots, t_N .

Here \dim_{aff} is *affinity dimension*—an symbolic upper bound for \dim_{H} .

Feng–Xie (2025) and Morris–Sert (2025). For **reducible** $X \subset \mathbb{R}^d$ such that some proper A^\top -invariant subspace W has $\dim_{\text{aff}}(A^\top|_W) < \min\{1, \dim_{\text{aff}}(A)\}$, there is $V \in \mathbb{R}\mathbb{P}^{d-1}$ with $V \subseteq W$ and

$$\overline{\dim}_{\text{M}}(\text{proj}_V(X)) < \min\{1, \dim_{\text{H}}(X)\}$$

for **Lebesgue-almost every** choice of translations t_1, \dots, t_N .

Question. Which self-affine sets $X \subset \mathbb{R}^d$ satisfy $\mathcal{E}(X) = \emptyset$ in every dimension d , for every choice of translations t_1, \dots, t_N , and without separation assumptions?

Corollaries of our main theorem

Although not stated explicitly in Hochman–Rapaport (2021), the following result follows from their work via the arguments of Bárány–Hochman–Rapaport (2019).

Theorem (Hochman–Rapaport (2021)). *Let $X \subset \mathbb{R}^2$ be a proximal, strongly irreducible self-affine set satisfying the exponential separation condition. Then for every $V \in \mathbb{R}P^1$,*

$$\dim_{\mathrm{H}}(\mathrm{proj}_V(X)) = \min\{1, \dim_{\mathrm{H}}(X)\}.$$

A corollary of our main theorem removes the exponential separation hypothesis:

Corollary 1. *Let $X \subset \mathbb{R}^2$ be a proximal, strongly irreducible self-affine set. Then for every $V \in \mathbb{R}P^1$,*

$$\dim_{\mathrm{H}}(\mathrm{proj}_V(X)) = \min\{1, \dim_{\mathrm{H}}(X)\}.$$

Corollaries of our main theorem

Here $\dim_{\text{M}}(X)$ is the *Minkowski dimension*—the growth rate of the number of δ -balls needed to cover X as $\delta \rightarrow 0$, always $\geq \dim_{\text{H}}(X)$ which allows covers by sets of varying radii.

Theorem (Hochman–Rapaport (2021)). *Let $X \subset \mathbb{R}^2$ be a proximal, strongly irreducible self-affine set satisfying the exponential separation condition. Then*

$$\dim_{\text{H}}(X) = \dim_{\text{M}}(X) = \dim_{\text{aff}}(X).$$

A corollary of our main theorem removes the exponential separation hypothesis when the upper Minkowski dimension is at most one:

Corollary 2. *Let $X \subset \mathbb{R}^2$ be a proximal, strongly irreducible self-affine set with $\overline{\dim}_{\text{M}}(X) \leq 1$. Then*

$$\dim_{\text{H}}(X) = \dim_{\text{M}}(X).$$

Main Theorem

Main Theorem. Let $X \subset \mathbb{R}^d$ be a *Zariski dense self-affine set*. Then, for every $V \in \mathbb{R}\mathbb{P}^{d-1}$,

$$\dim_{\text{H}}(\text{proj}_V(X)) = \min\{1, \dim_{\text{H}}(X)\} = \min\{1, \overline{\dim}_{\text{M}}(X)\}.$$

- ▶ **Corollary 1** follows once we show that proximality and strong irreducibility imply Zariski density for $A \in \text{GL}(2, \mathbb{R})^N$.
- ▶ **Corollary 2** then follows trivially from

$$\begin{aligned} \underline{\dim}_{\text{M}}(X) &\geq \dim_{\text{H}}(X) \geq \dim_{\text{H}}(\text{proj}_V(X)) \\ &= \min\{1, \overline{\dim}_{\text{M}}(X)\} = \overline{\dim}_{\text{M}}(X). \end{aligned}$$

It also provides a **partial positive answer** to the folklore question:

Question (Folklore). Does the Minkowski dimension of every self-affine set exist?

Zariski density

Let $\mathcal{S}(A) = \langle A_1, \dots, A_N \rangle$ be the subsemigroup generated by A .

Definition. A tuple $A = (A_1, \dots, A_N) \in \text{GL}(d, \mathbb{R})^N$ is **Zariski dense** if the Zariski closure of $\mathcal{S}(A)$ in $\text{GL}(d, \mathbb{R})$ is all of $\text{GL}(d, \mathbb{R})$.

Equivalently, every polynomial in the matrix entries and $1/\det$ that vanishes on $\mathcal{S}(A)$ vanishes on all of $\text{GL}(d, \mathbb{R})$: the matrices satisfy no nontrivial polynomial relation, and the semigroup is algebraically as large as possible.

Two consequences. If A is Zariski dense, then

- 1** A is strongly k -irreducible for every $k \in \{1, \dots, d-1\}$,
- 2** A is strongly pinching.

Both lift the strong irreducibility and proximality of A to the exterior powers $\wedge^k \mathbb{R}^d$.

Strong k -irreducibility and strong pinching

For $k \in \{1, \dots, d-1\}$, each $A \in \text{GL}(d, \mathbb{R})$ induces a linear map $A^{\wedge k}$ on the exterior power $\wedge^k \mathbb{R}^d$, whose eigenvalues are the k -fold products of distinct eigenvalues of A . Intuitively, $A^{\wedge k}$ records how A distorts k -dimensional parallelepipeds.

Definition. We say that $A \in \text{GL}(d, \mathbb{R})^N$ is

- ▶ *strongly k -irreducible* if no finite union of proper non-trivial subspaces of $\wedge^k \mathbb{R}^d$ is invariant under every $A_i^{\wedge k}$,
- ▶ *strongly pinching* if some single $A \in \mathcal{S}(A)$ has every $A^{\wedge k}$ diagonalizable with eigenvalues of pairwise distinct moduli.

Strong 1-irreducibility recovers strong irreducibility. **Strong pinching sharpens proximality:** distinct moduli force a simple dominant eigenvalue, so a single $A^{\wedge k}$ witnesses proximality on every exterior power $\wedge^k \mathbb{R}^d$ at once.

When does Zariski density hold?

The following proposition verifies Corollaries 1 and 2.

Proposition. *For contractive $A \in \mathrm{GL}(2, \mathbb{R})^N$, the tuple A is Zariski dense if and only if A is 1-proximal and strongly 1-irreducible.*

- ▶ In $d = 2$, the proper algebraic subgroups of $\mathrm{PGL}(2, \mathbb{R})$ are few—diagonal, upper triangular, $\mathrm{SO}(2)$, and finite groups—and proximality together with strong irreducibility rules each out, while contractivity supplies the missing determinant information. Without contractivity the equivalence fails in every d : $\mathrm{SL}(d, \mathbb{R})$ is a proper algebraic subgroup of $\mathrm{GL}(d, \mathbb{R})$.
- ▶ In $d \geq 3$ the equivalence breaks: there exist contractive tuples that are strongly pinching and strongly k -irreducible for every $k \in \{1, \dots, d - 1\}$, yet preserve an indefinite quadratic form and are confined to its orthogonal group up to scaling. Hence Zariski density is **strictly stronger** than strong pinching plus strong k -irreducibility in dimension $d \geq 3$.

When does Zariski density hold?

Proposition. Zariski dense tuples form an *open and dense* subset of $GL(d, \mathbb{R})^N$.

A tuple fails to be Zariski dense only when its semigroup is confined to a proper algebraic subgroup of $GL(d, \mathbb{R})$, such as one preserving a quadratic form or a subspace, or with determinants in $\{1, -1\}$.

Sharpness. Feng–Xie (2025) and Morris–Sert (2025) construct reducible self-affine sets with non-trivial exceptional set at Lebesgue-typical translations, so the conclusion of Main Theorem genuinely requires Zariski density. Whether it is the *minimal* algebraic hypothesis—in particular, whether e.g. irreducible quadratic-form-preserving tuples in $d \geq 3$ admit an all-directions theorem—remains open.

Let us next sketch the proof of the main theorem.

Step 1: the measure theorem (the engine)

- ▶ The proof of the main theorem routes through self-affine measures and reduces to subsystems. The first step is the measure analogue.
- ▶ A fully supported Bernoulli measure ν on $\mathcal{I}^{\mathbb{N}} = \{1, \dots, N\}^{\mathbb{N}}$ pushes forward through the natural coding $\pi: \mathcal{I}^{\mathbb{N}} \rightarrow X$ to a *self-affine measure* $\mu = \pi_*\nu$. Its *Lyapunov dimension* $\dim_L(\nu)$ is determined by the Kolmogorov–Sinai entropy of ν and the Lyapunov exponents of A .

Theorem 3. *Let ν be a fully supported Bernoulli measure and μ the associated self-affine measure. Suppose Φ satisfies the *exponential separation condition* and A is *k-proximal* and *strongly k-irreducible* for every $k \in \{1, \dots, d-1\}$. Then, for every $V \in \mathbb{RP}^{d-1}$,*

$$\dim((\text{proj}_V)_*\mu) = \min\{1, \dim_L(\nu)\}.$$

Let us next define the symbolic dimensions $\dim_{\text{aff}}(A)$ and $\dim_L(\nu)$.

Symbolic dimensions

To make $\dim_{\text{aff}}(A)$ and $\dim_L(\nu)$ precise, we use Falconer's *singular value function*. For $A \in \text{GL}(d, \mathbb{R})$, let $\alpha_1(A) \geq \dots \geq \alpha_d(A) > 0$ be the singular values—the semi-axes of the ellipsoid $A(B(0, 1))$ —and for $0 \leq s \leq d$ with $k = \lfloor s \rfloor$, set

$$\varphi^s(A) = \alpha_1(A) \cdots \alpha_k(A) \alpha_{k+1}(A)^{s-k}.$$

The *affinity dimension* $\dim_{\text{aff}}(A)$ is the unique $s \in [0, d]$ with

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\mathbf{i} \in \mathcal{I}^n} \varphi^s(A_{\mathbf{i}}) = 0,$$

or d if no such s exists, and depends only on Φ . For a Bernoulli measure ν on $\mathcal{I}^{\mathbb{N}}$, the *Lyapunov dimension* $\dim_L(\nu)$ is the unique s with $h(\nu) + \Lambda(A, \nu, s) = 0$, where $h(\nu)$ is the entropy of ν and

$$\Lambda(A, \nu, s) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \varphi^s(A_{\mathbf{i}|_n}) d\nu(\mathbf{i}).$$

Both bound the geometric dimension from above: Falconer (1988) proved $\overline{\dim}_M(X) \leq \dim_{\text{aff}}(A)$, and always $\dim_L(\nu) \leq \dim_{\text{aff}}(A)$.

Step 2: subsystems

Theorem 3 needs **exponential separation** and **k -proximality**, **strong k -irreducibility** for every k . The main theorem assumes neither. We extract a subsystem $\Phi' \subseteq \Phi$ that has both.

Proposition 4. *For every $\varepsilon > 0$ there is a finite collection of words $\mathcal{J} \subset \mathcal{I}^*$ such that $\Phi' = (\varphi_i)_{i \in \mathcal{J}}$*

- ▶ satisfies the **strong separation condition** (hence exponential separation),
- ▶ is **dominated**: $\|A_{ij}\| \geq \kappa \|A_i\| \|A_j\|$ on \mathcal{J}^* for some $\kappa \in (0, 1]$,
- ▶ has **k -proximal** and **strongly k -irreducible** for every k ,
- ▶ has $\dim_{\text{aff}}(A') \geq \min\{1, \overline{\dim}_M(X)\} - \varepsilon$.

The proof of Proposition 4 proceeds in **two steps**: first we construct a dominated subsystem, then a strongly separated packing inside it.

Step 3: transfer to sets

Fix $\varepsilon > 0$ small, take Φ' and \mathcal{J} from [Proposition 4](#), and set

$$s = \min\{1, \dim_{\text{aff}}(A')\} - \varepsilon \in (0, 1).$$

Choose $n \geq 1$ with $\kappa^s \sum_{i \in \mathcal{J}^n} \|A_i\|^s > 1$, set $\mathcal{K} = \mathcal{J}^n$ and $\Phi'' = (\varphi_i)_{i \in \mathcal{K}}$, and let ν be the Bernoulli measure on $\mathcal{K}^{\mathbb{N}}$ with weights $p_i \approx \|A_i\|^s$. The super-multiplicativity from domination then yields

$$\dim_{\text{L}}(\nu) \geq s.$$

Since $\mathcal{K} = \mathcal{J}^n$, the subsystem Φ'' inherits the strong separation condition from Φ' and is k -proximal and strongly k -irreducible for every k . By [Theorem 3](#),

$$\begin{aligned} \dim_{\text{H}}(\text{proj}_V(X)) &\geq \dim((\text{proj}_V)_* \mu) = \min\{1, \dim_{\text{L}}(\nu)\} \\ &\geq \min\{1, \dim_{\text{aff}}(A')\} - \varepsilon \geq \min\{1, \overline{\dim}_{\text{M}}(X)\} - 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ finishes the proof of the Main Theorem assuming [Steps 1 and 2](#). □

Step 1: proof outline

We now sketch the proof of Theorem 3, the measure theorem of [Step 1](#). The argument proceeds in two steps.

Step 1.1 (Furstenberg-typical directions). We establish the conclusion at directions V typical for a natural [stationary measure](#) μ_F on \mathbb{RP}^{d-1} associated with A^\top and ν , by combining the entropy-dimension formula of Rapaport (2024) with Feng's (2023) exact-dimensionality theorem.

Step 1.2 (upgrade to every direction). We pass from μ_F -typical V to every prescribed $V \in \mathbb{RP}^{d-1}$. The catch is that μ_F can be singular with respect to the spherical measure, with Cantor-like support missing the directions we want. A suspension semiflow then transports the typical-direction estimate to [any](#) such V .

Exponential separation

The measure theorem requires a quantitative no-overlap condition.

Definition. We say that $\Phi = (\varphi_1, \dots, \varphi_N)$ satisfies the *exponential separation condition* if there exist $c > 0$ and infinitely many $n \geq 1$ such that for all distinct $i \neq j \in \mathcal{I}^n$,

$$\|\varphi_i - \varphi_j\|_\infty \geq c^n.$$

Exponential separation is the standard hypothesis in the literature for dimension theorems on self-affine measures. Even for self-similar sets on the real line, dimension theory without it is wide open—this is the celebrated *dimension drop conjecture*.

Intuition. Exponential separation is mild: it holds for any IFS with algebraic entries and no exact overlaps (Gamburd–Jakobson–Sarnak (1999) for $d = 2$, generalized here to all d), and in \mathbb{R} outside of zero Hausdorff dimensional exceptional set (Hochman (2014)).

The Furstenberg measure

When A is 1-proximal and strongly 1-irreducible, there is a unique probability measure μ_F on \mathbb{RP}^{d-1} , the *Furstenberg measure*, satisfying the stationarity equation

$$\mu_F = \sum_{i=1}^N p_i (A_i^\top)_* \mu_F, \quad p_i = \nu([i]).$$

Intuition. Consider the random walk on \mathbb{RP}^{d-1} that, at each step, draws $i \in \mathcal{I}$ from $(p_i)_i$ and applies A_i^\top to the current direction. The Furstenberg measure μ_F is the unique stationary distribution of this walk, and iterates of any starting direction converge in law to μ_F (Furstenberg (1963)).

Why A^\top and not A ? Pulling V back through $\text{proj}_V \circ \varphi_i$ gives $A_i^\top V$, since $\langle A_i x, V \rangle = \langle x, A_i^\top V \rangle$. Hence the natural action on projection directions is through the transposed semigroup.

Step 1.1: Furstenberg-typical projection

Theorem 5 (Rapaport (2024) and Feng (2023)). *Let ν be a fully supported Bernoulli measure and μ the associated self-affine measure. Suppose Φ satisfies the *exponential separation condition* and A is k -proximal and strongly k -irreducible for every $k \in \{1, \dots, d-1\}$. Then, μ_F -almost every $V \in \mathbb{R}\mathbb{P}^{d-1}$,*

$$\dim((\text{proj}_V)_*\mu) = \min\{1, \dim_L(\nu)\}.$$

The proof of Theorem 5 combines two ingredients.

- ▶ Rapaport (2024) supplies the *entropy-dimension* identity for $(\text{proj}_V)_*\mu$ at every $V \in \mathbb{R}\mathbb{P}^{d-1}$.
- ▶ Feng (2023) upgrades the entropy dimension to *exact dimension* (Hausdorff = packing = entropy) on a set of full μ_F -measure.

Combining the two yields the conclusion at a μ_F -typical V —this is **Step 1.1**.

Step 1.2: from μ_F -typical to every direction

Why is **Theorem 5** not enough? We need an estimate at a **prescribed** V , not a typical one. The Furstenberg measure μ_F can be singular, even supported on a Cantor-like subset of \mathbb{RP}^{d-1} that misses most directions.

Step 1.2.1 (suspension semiflow). Build a **suspension semiflow** on $\mathcal{I}^{\mathbb{N}} \times \mathbb{RP}^{d-1}$ whose direction coordinate **equidistributes** to μ_F from any starting V .

Step 1.2.2 (local entropy averages). Verify the **local-entropy-averages hypothesis** of Hochman–Shmerkin (2012) at V , transporting Rapaport’s typical-direction estimate.

Step 1.2.3 (matching upper bound). Combine with Rossi’s (2014) **upper bound** to upgrade the lower bound to exact dimension.

Step 1.2.1: suspension semiflow

Define a base map and a roof function on $\mathcal{I}^{\mathbb{N}} \times \mathbb{RP}^{d-1}$ by

$$F(\mathbf{i}, V) = (\sigma \mathbf{i}, A_{i_1}^{\top} V), \quad r(\mathbf{i}, V) = -\log_2 \|A_{i_1}^{\top} V\|.$$

Identifying $(\mathbf{i}, V, r(\mathbf{i}, V)) \sim (F(\mathbf{i}, V), 0)$, the suspension space is

$$\mathcal{X} = \{(\mathbf{i}, V, t) \in \mathcal{I}^{\mathbb{N}} \times \mathbb{RP}^{d-1} \times \mathbb{R} : 0 \leq t < r(\mathbf{i}, V)\},$$

and the semiflow T_t increases the height t and folds at the roof.

Equidistribution along orbits. Since μ_F is F -invariant, the normalized restriction

$$m = \frac{(\nu \times \mu_F \times \mathcal{L}^1)|_{\mathcal{X}}}{(\nu \times \mu_F \times \mathcal{L}^1)(\mathcal{X})}$$

is a T_t -invariant Borel probability on \mathcal{X} . Along T_t -orbits, the direction coordinate converges in law to μ_F from **any** starting $V \in \mathbb{RP}^{d-1}$.

Step 1.2.2: local entropy averages

Theorem (Hochman–Shmerkin (2012)). *Let ν be a Borel probability on $\mathcal{I}^{\mathbb{N}}$, $f: \mathcal{I}^{\mathbb{N}} \rightarrow [-1, 1]$ continuous, and $\{\mathcal{Q}_n^q\}$ a refining family of partitions of $\mathcal{I}^{\mathbb{N}}$ with $\text{diam}(f([i])) \approx 2^{-qn}$ on \mathcal{Q}_n^q . If, for ν -almost every i ,*

$$\limsup_{q \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{1}{qn \log 2} \sum_{i=1}^n H(f_* \nu|_{[\mathcal{Q}_i^q(i)]}, \mathcal{D}_{q(i+1)}) \geq \alpha,$$

then $\underline{\dim}_H(f_ \nu) \geq \alpha$.*

Apply this with $f = \text{proj}_V \circ \pi$ and cutsets matched to $\|A_i^\top | V\| \approx 2^{-qi}$. The flow's T_q -steps match these cutset depths up to a bounded error, so [Rapaport's entropy bound](#) from [Step 1.1](#) verifies the local-entropy-averages hypothesis at the evolving direction; Furstenberg comparability then transfers it to the prescribed V , giving for every $V \in \mathbb{RP}^{d-1}$,

$$\underline{\dim}_H((\text{proj}_V)_* \mu) \geq \min\{1, \dim_L(\nu)\}.$$

Step 1.2.3: matching upper bound

Proposition (Rossi (2014)). *For any ergodic shift-invariant Borel probability ν on $\mathcal{I}^{\mathbb{N}}$ and the associated self-affine measure $\mu = \pi_*\nu$,*

$$\overline{\dim}_p(\mu) \leq \min\{d, \dim_L(\nu)\}.$$

The projection $\text{proj}_V: \mathbb{R}^d \rightarrow V$ is Lipschitz and lands on a line, so

$$\overline{\dim}_p((\text{proj}_V)_*\mu) \leq \min\{1, \overline{\dim}_p(\mu)\} \leq \min\{1, \dim_L(\nu)\}.$$

Together with the [Step 1.2.2](#) lower bound, for every $V \in \mathbb{RP}^{d-1}$ we obtain

$$\dim((\text{proj}_V)_*\mu) = \min\{1, \dim_L(\nu)\},$$

upgrading [Theorem 5](#) from μ_F -typical to all directions and completing [Step 1](#).

Step 2: from measures to sets

Step 1 produced the **measure theorem**: under **exponential separation** and **k -proximity** plus **strong k -irreducibility** for every k ,

$$\dim((\text{proj}_V)_*\mu) = \min\{1, \dim_L(\nu)\}$$

at every $V \in \mathbb{R}P^{d-1}$. The Main Theorem asks the same conclusion for $\dim_H(\text{proj}_V(X))$ from **Zariski density alone**—no separation, no algebraic hypotheses on A beyond Zariski density itself.

Strategy. Approximate X from inside by a subsystem $X' \subseteq X$ such that Φ' satisfies the **strong separation condition**, A' is **k -proximal** and **strongly k -irreducible** for every k , and $\dim_{\text{aff}}(A') \geq \min\{1, \overline{\dim}_M(X)\} - \varepsilon$. Apply **Step 1** to a self-affine measure on X' with $\dim_L(\nu)$ close to $\dim_{\text{aff}}(A')$.

The subsystem is built in two stages—**Step 2.1** finds a dominated subsystem and **Step 2.2** makes it strongly separated.

Step 2.1: dominated subsystem

Proposition. Let $A \in GL(d, \mathbb{R})^N$ be 1-proximal and strongly 1-irreducible, and fix $i \in \mathcal{I}^*$ with A_i proximal. There exist $K \geq 1$ and maps $\bar{h}, \bar{k}: \mathcal{I}^* \rightarrow \bigcup_{m=0}^K \mathcal{I}^m$ such that, for every finite $\mathcal{D} \subset \mathcal{I}^*$, the sandwiched tuple

$$A' = (A_{\bar{h}(j)} A_j A_{\bar{k}(j)})_{j \in \mathcal{D}}$$

is dominated.

This extends Bárány–Jordan–K–Rams (2021) from $d = 2$ to all d .

Strategy. Take the cone $\mathcal{C} = B(V, \varepsilon)$ around the leading eigenline V of A_i , with \mathcal{C} disjoint from the repelling hyperplane W . For each j , the suffix $\bar{k}(j)$ steers the image of $A_j A_{\bar{k}(j)} \mathcal{C}$ into one of finitely many cover sets Q_i of $\mathbb{R}P^{d-1}$; the prefix $\bar{h}(j)$ then uses an irreducibility word to push Q_i off W and powers of A_i to contract back into \mathcal{C} . Hence \mathcal{C} is a **strongly invariant multicone** for A' , and domination follows from Bochi–Gourmelon (2009).

Step 2.2: strongly separated subsystem

Proposition. Let $A \in GL(d, \mathbb{R})^N$ be Zariski dense. For every $\varepsilon > 0$ there is a finite $\mathcal{J} \subset \mathcal{I}^*$ such that

- ▶ $\Phi' = (\varphi_j)_{j \in \mathcal{J}}$ satisfies the *strong separation condition*;
- ▶ $A' = (A_j)_{j \in \mathcal{J}}$ is dominated, and *k-proximal and strongly k-irreducible* for every $k \in \{1, \dots, d-1\}$;
- ▶ $\min\{1, \overline{\dim}_M(X)\} - \varepsilon \leq \dim_{\text{aff}}(A') < d$.

The proof combines three ingredients:

- ▶ **Packing** yields $\gtrsim r^{-\overline{\dim}_M(X)+\varepsilon}$ cylinders of comparable size at scale r with pairwise disjoint images.
- ▶ **Step 2.1** sandwiches each cylinder so that A' is dominated.
- ▶ **Zariski density** yields a **strongly pinching** word, which gives k -proximality, and **subspace avoidance** in every exterior power, which gives strong k -irreducibility.

Finishing the proof

Steps 2.1 and 2.2 together prove Proposition 4, completing Step 2. As sketched earlier in Step 3, plugging strongly separated dominated subsystems of Proposition 4 into the measure projection theorem, Theorem 3, then yields

$$\dim_{\text{H}}(\text{proj}_V(X)) \geq \min\{1, \overline{\dim}_{\text{M}}(X)\} - 2\varepsilon$$

for every $V \in \mathbb{R}\mathbb{P}^{d-1}$. Letting $\varepsilon \rightarrow 0$ proves the Main Theorem:

Main Theorem. *Let $X \subset \mathbb{R}^d$ be a Zariski dense self-affine set. Then, for every $V \in \mathbb{R}\mathbb{P}^{d-1}$,*

$$\dim_{\text{H}}(\text{proj}_V(X)) = \min\{1, \dim_{\text{H}}(X)\} = \min\{1, \overline{\dim}_{\text{M}}(X)\}.$$